

FUSION PROCEDURE FOR CYCLOTOMIC BMW ALGEBRAS

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ABSTRACT. Inspired by the work [IMog2], in this note, we prove that the pairwise orthogonal primitive idempotents of generic cyclotomic Birman-Murakami-Wenzl algebras can be constructed by consecutive evaluations of a certain rational function. In the appendix, we prove a similar result for generic cyclotomic Nazarov-Wenzl algebras.

1. INTRODUCTION

1.1. The primitive idempotents of a symmetric group \mathfrak{S}_n , showed by Jucys [Juc], can be obtained by taking a certain limiting process on a rational function. The process is now commonly known as the fusion procedure, which has been further developed in the situation of Hecke algebras [Ch]; see also [Na2-4]. In [Mo], Molev has presented another approach of the fusion procedure for \mathfrak{S}_n , which depends on the existence of a maximal commutative subalgebra generated by the Jucys-Murphy elements. In his approach, the primitive idempotents are obtained by consecutive evaluations of a certain rational function. The new version of the fusion procedure has been generalized to the Hecke algebras of type A [IMO], to the Brauer algebras [IM, IMog1], to the Birman-Murakami-Wenzl algebras [IMog2], to the complex reflection groups of type $G(d, 1, n)$ [OgPA1], to the Ariki-Koike algebras [OgPA2], to the wreath products of finite groups by the symmetric group [PA], to the degenerate cyclotomic Hecke algebras [ZL], to the Yokonuma-Hecke algebras [C1], to the cyclotomic Yokonuma-Hecke algebras [C2, Appendix] and to the degenerate cyclotomic Yokonuma-Hecke algebras [C3].

1.2. The Birman-Murakami-Wenzl (for brevity, BMW) algebra was algebraically defined by Birman and Wenzl [BW], and independently Murakami [Mu], which is an algebra generated by some elements satisfying certain particular relations. These relations are in fact implicitly modeled on the ones of certain algebra of tangles studied by Kauffman [Ka] and Morton and Traczyk [MT], which is known as a Kauffman tangle algebra. BMW algebra are closely related to Artin braid groups of type A , Iwahori-Hecke algebras of type A , quantum groups, Brauer algebras and other diagram algebras; see [Eny1-2, HuXi, Hu, LeRa, MW, RuSi4-6, RuSo, Xi] and the references therein.

Motivated by studying link invariants, Häring-Oldenburg [HO] introduced a class of finite dimensional associative algebras called cyclotomic Birman-Murakami-Wenzl (for brevity, BMW) algebras, generalizing the notions of BMW algebras. Such algebras are closely related to Artin braid groups of type B , cyclotomic Hecke algebras and other research objects, and have been studied by a lot of authors from different perspectives; see [Go1-4, GoHM1-2, HO, OrRa, RuSi2-3, RuXu, Si, WiYu1-3, Xu, Yu] and so on.

1.3. Inspired by the work [IMog2] on the fusion procedures of BMW algebras, in this note we prove that a complete set of pairwise orthogonal primitive idempotents of cyclotomic BMW algebras can be derived by consecutive evaluations of a certain rational function in several variables. In the appendix, we prove a similar result for generic cyclotomic Nazarov-Wenzl algebras.

This paper is organized as follows. In Section 2, we recall some preliminaries and introduce the primitive idempotents $E_{\mathcal{T}}$ of cyclotomic BMW algebras. In Section 3, we establish the fusion formula for the primitive idempotent $E_{\mathcal{T}}$. In Section 4 (Appendix), we develop the fusion formulas for the primitive idempotents of cyclotomic Nazarov-Wenzl algebras.

2. PRELIMINARIES

2.1. Cyclotomic Birman-Murakami-Wenzl algebras.

Definition 2.1. Assume that \mathbb{K} is an algebraically closed field containing $\delta_j, 0 \leq j \leq d-1$, and some nonzero elements $\rho, q, q - q^{-1}$ and $v_i, 1 \leq i \leq d$, and that they satisfy the relation $\rho - \rho^{-1} = (q - q^{-1})(\delta_0 - 1)$.

Fix $n \geq 1$. The cyclotomic Birman-Murakami-Wenzl algebra $\mathcal{B}_{d,n}$ is the \mathbb{K} -algebra generated by the elements $X_1^{\pm 1}, T_i^{\pm 1}$ and E_i ($1 \leq i \leq n-1$) with the following relations:

- (1) (Inverses) $T_i T_i^{-1} = T_i^{-1} T_i = 1$ and $X_1 X_1^{-1} = X_1^{-1} X_1 = 1$.
- (2) (Idempotent relations) $E_i^2 = \delta_0 E_i$ for $1 \leq i \leq n-1$.
- (3) (Affine braid relations)
 - (a) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ if $|i - j| \geq 2$.
 - (b) $X_1 T_1 X_1 T_1 = T_1 X_1 T_1 X_1$ and $X_1 T_j = T_j X_1$ if $j \geq 2$.
- (4) (Tangle relations)
 - (a) $E_i E_{i\pm 1} E_i = E_i$.
 - (b) $T_i T_{i\pm 1} E_i = E_{i\pm 1} E_i$ and $E_i T_{i\pm 1} T_i = E_i E_{i\pm 1}$.
 - (c) For $1 \leq j \leq d-1$, $E_1 X_1^j E_1 = \delta_j E_1$.
- (5) (Kauffman skein relations) $T_i - T_i^{-1} = (q - q^{-1})(1 - E_i)$ for $1 \leq i \leq n-1$.
- (6) (Untwisting relations) $T_i E_i = E_i T_i = \rho^{-1} E_i$ for $1 \leq i \leq n-1$.
- (7) (Unwrapping relations) $E_1 X_1 T_1 X_1 = \rho E_1 = X_1 T_1 X_1 E_1$.
- (8) (Cyclotomic relation) $(X_1 - v_1)(X_1 - v_2) \cdots (X_1 - v_d) = 0$.

In $\mathcal{B}_{d,n}$, We define inductively the following elements:

$$X_{i+1} := T_i X_i T_i \quad \text{for } i = 1, \dots, n-1. \quad (2.1)$$

It can be easily checked that the elements X_1, \dots, X_n commute with each other, and moreover, we have

$$E_i X_i X_{i+1} = X_i X_{i+1} E_i = E_i \quad \text{for } i = 1, \dots, n-1. \quad (2.2)$$

We now define the following elements (see [IMog2, (2.15)]):

$$T_i(u, v) = T_i + \frac{(q - q^{-1})u}{v - u} + \frac{(q - q^{-1})u}{u + \rho q v} E_i \quad \text{for } i = 1, \dots, n-1. \quad (2.3)$$

Note that $E_i^2 = \delta_0 E_i$, where $\delta_0 = \frac{(q^{-1} + \rho^{-1})(\rho q - 1)}{q - q^{-1}}$. By using this, it can be easily checked that (see [IMOG2, (2.17-18)])

$$T_i(u, v)T_i(v, u) = f(u, v) \quad \text{for } i = 1, \dots, n-1, \quad (2.4)$$

where

$$f(u, v) = f(v, u) = \frac{(u - q^2 v)(u - q^{-2} v)}{(u - v)^2}. \quad (2.5)$$

2.2. Combinatorics. $\lambda = (\lambda_1, \dots, \lambda_k)$ is called a partition of n if it is a finite sequence of weakly decreasing nonnegative integers whose sum is n . We set $|\lambda| := n$. We shall identify a partition λ with a Young diagram, which is the set

$$[\lambda] := \{(i, j) \mid i \geq 1 \text{ and } 1 \leq j \leq \lambda_i\}.$$

We shall regard λ as a left-justified array of boxes such that there exist λ_j boxes in the j -th row for $j = 1, \dots, k$. We write $\theta = (a, b)$ if the box θ lies in row a and column b .

Similarly, a d -partition of n is an ordered d -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ of partitions $\lambda^{(k)}$ such that $\sum_{k=1}^d |\lambda^{(k)}| = n$. We denote by $\mathcal{P}_d(n)$ the set of d -partitions of n . We shall identify a d -partition λ with its Young diagram, which is the ordered d -tuple of the Young diagrams of its components. We write $\theta = (\theta, s)$ if the box θ lies in the component $\lambda^{(s)}$.

Assume that λ and μ are two d -partitions. We say that λ is obtained from μ by adding a box if there exists a pair (j, t) such that $\lambda_j^{(t)} = \mu_j^{(t)} + 1$ and $\lambda_i^{(s)} = \mu_i^{(s)}$ for $(i, s) \neq (j, t)$. In this case, we will also say that μ is obtained from λ by removing a box.

Set

$$\Lambda_{d,n}^+ := \{(l, \lambda) \mid 0 \leq l \leq \lfloor n/2 \rfloor, \lambda \in \mathcal{P}_d(n - 2l)\}.$$

The combinatorial objects appearing in the representation theory of $\mathcal{B}_{d,n}$ will be up-down tableaux. For $(f, \lambda) \in \Lambda_{d,n}^+$, an n -updown λ -tableau, or more simply an updown λ -tableau, is a sequence $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n)$ of d -partitions such that $\mathcal{T}_n = \lambda$ and \mathcal{T}_i is obtained from \mathcal{T}_{i-1} by either adding or removing a box, for $i = 1, \dots, n$, where we set $\mathcal{T}_0 = \emptyset$. Let $\mathcal{T}_n^{ud}(\lambda)$ be the set of updown λ -tableaux of n .

Suppose that $(f, \lambda) \in \Lambda_{d,n}^+$ and $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_n) \in \mathcal{T}_n^{ud}(\lambda)$. Let

$$c(\mathcal{U}|k) = \begin{cases} v_s q^{2(j-i)} & \text{if } \mathcal{U}_k = \mathcal{U}_{k-1} \cup ((i, j), s), \\ v_s^{-1} q^{2(i-j)} & \text{if } \mathcal{U}_{k-1} = \mathcal{U}_k \cup ((i, j), s). \end{cases} \quad (2.6)$$

Given a box $\alpha = ((i, j), s)$, we define the content of it by

$$c(\mathcal{U}|\alpha) = \begin{cases} v_s q^{2(j-i)} & \text{if } \alpha \text{ is an addable box of } \mathcal{U}, \\ v_s^{-1} q^{2(i-j)} & \text{if } \alpha \text{ is a removable box of } \mathcal{U}. \end{cases} \quad (2.7)$$

We give the generalizations of some constructions in [IM, Section 3]. Suppose that $(f, \lambda) \in \Lambda_{d,n}^+$ and $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_n) \in \mathcal{T}_n^{ud}(\lambda)$. Set $\mu = \mathcal{T}_{n-1}$ and consider the updown μ -tableau $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_{n-1})$. We now define two d -tuples of infinite matrices

$$M(\mathcal{U}) = (m_1(\mathcal{U}), \dots, m_d(\mathcal{U})) \quad \text{and} \quad \overline{M}(\mathcal{U}) = (\overline{m}_1(\mathcal{U}), \dots, \overline{m}_d(\mathcal{U})),$$

here the rows and columns of each $m_s(\mathcal{U})$ or $\overline{m}_s(\mathcal{U})$ are labelled by positive integers and only a finite number of entries in each of the matrices are nonzero. The entry m_{ij}^s of the

matrix $m_s(\mathcal{U})$ (respectively, the entry \overline{m}_{ij}^s of the matrix $\overline{m}_s(\mathcal{U})$) equals the number of times that the box $((i, j), s)$ is added (respectively, removed) in the sequence $(\emptyset, \mathcal{T}_1, \dots, \mathcal{T}_{n-1})$.

For each $k \in \mathbb{Z}$ and $1 \leq s \leq d$, we define two nonnegative integers $d_k^s = d_k(m_s(\mathcal{U}))$ and $\overline{d}_k^s = d_k(\overline{m}_s(\mathcal{U}))$ as the sums of the entries of the matrices $m_s(\mathcal{U})$ and $\overline{m}_s(\mathcal{U})$ on the k -th diagonal, that is,

$$d_k^s = \sum_{j-i=k} m_{ij}^s \quad \text{and} \quad \overline{d}_k^s = \sum_{j-i=k} \overline{m}_{ij}^s. \quad (2.8)$$

Furthermore, we define the indexes $g_k^s = g_k(m_s(\mathcal{U}))$ and $\overline{g}_k^s = g_k(\overline{m}_s(\mathcal{U}))$ as follows:

$$g_k^s = \delta_{k0} + d_{k-1}^s + d_{k+1}^s - 2d_k^s \quad \text{and} \quad \overline{g}_k^s = \overline{d}_{k-1}^s + \overline{d}_{k+1}^s - 2\overline{d}_k^s. \quad (2.9)$$

Finally, we define some integer p_1, \dots, p_n associated to \mathcal{T} inductively such that p_k depends only on the first k d -partitions $(\mathcal{T}_1, \dots, \mathcal{T}_k)$ of \mathcal{T} . Therefore, it is enough to define p_n . We set

$$p_n = 1 - g_{k_n}(m_{s_n}(\mathcal{U})) \quad (2.10)$$

if \mathcal{T}_n is obtained from \mathcal{T}_{n-1} by adding a box $((i_n, j_n), s_n)$, where $k_n = j_n - i_n$;

$$p_n = 1 - g_{k'_n}(\overline{m}_{s'_n}(\mathcal{U})) \quad (2.11)$$

if \mathcal{T}_n is obtained from \mathcal{T}_{n-1} by removing a box $((i'_n, j'_n), s'_n)$, where $k'_n = j'_n - i'_n$.

Assume that $(f, \boldsymbol{\lambda}) \in \Lambda_{d,n}^+$, $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_n)$ is an n -updown $\boldsymbol{\lambda}$ -tableau and that $\mathcal{U} = (\mathcal{T}_1, \dots, \mathcal{T}_{n-1})$. We then define the element $f(\mathcal{T})$ inductively by

$$f(\mathcal{T}) = f(\mathcal{U})\varphi(\mathcal{U}, \mathcal{T}), \quad (2.12)$$

where

$$\varphi(\mathcal{U}, \mathcal{T}) = \prod_{\substack{k \neq k_n \\ k \in \mathbb{Z}}} (q^{2k_n} - q^{2k})^{g_k^{s_n}} \prod_{\substack{1 \leq t \leq d; t \neq s_n \\ k \in \mathbb{Z}}} (v_{s_n} q^{2k_n} - v_t q^{2k})^{g_k^t} \prod_{\substack{1 \leq r \leq d \\ k \in \mathbb{Z}}} (v_{s_n} q^{2k_n} - v_r^{-1} q^{-2k})^{\overline{g}_k^r}$$

if \mathcal{T}_n is obtained from \mathcal{T}_{n-1} by adding a box $((i_n, j_n), s_n)$, where $k_n = j_n - i_n$;

$$\varphi(\mathcal{U}, \mathcal{T}) = \prod_{\substack{k \neq k'_n \\ k \in \mathbb{Z}}} (q^{-2k'_n} - q^{-2k})^{\overline{g}_k^{s'_n}} \prod_{\substack{1 \leq t \leq d; t \neq s'_n \\ k \in \mathbb{Z}}} (v_{s'_n}^{-1} q^{-2k'_n} - v_t^{-1} q^{-2k})^{\overline{g}_k^t} \prod_{\substack{1 \leq r \leq d \\ k \in \mathbb{Z}}} (v_{s'_n}^{-1} q^{-2k'_n} - v_r q^{2k})^{g_k^r}$$

if \mathcal{T}_n is obtained from \mathcal{T}_{n-1} by removing a box $((i'_n, j'_n), s'_n)$, where $k'_n = j'_n - i'_n$.

In the special situation when $f = 0$, that is, $\boldsymbol{\lambda}$ is a d -partition of n , there is a natural bijection between the set of n -updown $\boldsymbol{\lambda}$ -tableaux and the set of standard $\boldsymbol{\lambda}$ -tableaux defined in [DJM, Definition (3.10)]. The following proposition is inspired by [IM, Proposition 3.3] and can be proved similarly.

Proposition 2.2. *If $\boldsymbol{\lambda}$ is a d -partition of n and $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_n)$ is an n -updown $\boldsymbol{\lambda}$ -tableau, then p_1, \dots, p_n are all equal to zero, and $f(\mathcal{T})$ is exactly equal to $F_{\boldsymbol{\lambda}}^{-1}$ defined in [OgPA2, Section 2.2(12)] when $d = m$.*

2.3. Idempotents of $\mathcal{B}_{d,n}$. Following [RuXu, Definition 3.4], we say that $\mathcal{B}_{d,n}$ is generic if the parameters v_i , $1 \leq i \leq d$, and q satisfy the conditions (1) the order $o(q^2)$ of q^2 satisfies $o(q^2) > 2n$; (2) $|r| \geq 2n$ whenever there exists $r \in \mathbb{Z}$ such that either $v_i v_j^{\pm 1} = q^{2r}$ for $i \neq j$, or $v_i = \pm q^r$. Following [WiYu1, Corollary 4.5], we say that $\mathcal{B}_{d,n}$ is admissible if the set $\{E_1, E_1 X_1, \dots, E_1 X_1^{d-1}\}$ is linearly independent in $\mathcal{B}_{d,2}$. It has been proved by Goodman [Go2, Theorem 4.4] that this admissible condition coincides with the \mathbf{u} -admissible condition defined in [RuXu, Definition 2.27].

From now on, we always assume that $\mathcal{B}_{d,n}$ is generic and admissible. Thus, by [RuXu, Lemma 3.5], we have $\mathcal{S} = \mathcal{T}$ if and only if $c(\mathcal{S}|k) = c(\mathcal{T}|k)$ for all $1 \leq k \leq n$. Therefore, the set $\{X_1, \dots, X_n\}$, as a family of JM-elements for $\mathcal{B}_{d,n}$ in the abstract sense defined in [Ma, Definition 2.4], satisfies the separation condition associated to the weakly cellular basis of $\mathcal{B}_{d,n}$ constructed in [RuXu, Theorem 4.19]. Note that the results in [Ma] also hold for $\mathcal{B}_{d,n}$ with respect to the weakly cellular basis. In particular, we can construct the primitive idempotents of $\mathcal{B}_{d,n}$ following the arguments in [Ma, Section 3].

For each $1 \leq k \leq n$, we define the following set:

$$\mathcal{R}(k) := \{c(\mathcal{S}|k) \mid \mathcal{S} \in \mathcal{T}_n^{ud}(\boldsymbol{\lambda}) \text{ for some } (f, \boldsymbol{\lambda}) \in \Lambda_{d,n}^+\}.$$

Suppose that $(f, \boldsymbol{\lambda}) \in \Lambda_{d,n}^+$ and $\mathcal{T} \in \mathcal{T}_n^{ud}(\boldsymbol{\lambda})$. We set

$$E_{\mathcal{T}} = \prod_{k=1}^n \left(\prod_{\substack{c \in \mathcal{R}(k) \\ c \neq c(\mathcal{T}|k)}} \frac{X_k - c}{c(\mathcal{T}|k) - c} \right). \quad (2.13)$$

By standard arguments in [Ma, Section 3], the elements $\{E_{\mathcal{T}} \mid \mathcal{T} \in \mathcal{T}_n^{ud}(\boldsymbol{\lambda}) \text{ for some } (f, \boldsymbol{\lambda}) \in \Lambda_{d,n}^+\}$ form a complete set of pairwise orthogonal primitive idempotents of $\mathcal{B}_{d,n}$. Moreover, the elements X_1, \dots, X_n generate a maximal commutative subalgebra of $\mathcal{B}_{d,n}$. We also have

$$X_k E_{\mathcal{T}} = E_{\mathcal{T}} X_k = c(\mathcal{T}|k) E_{\mathcal{T}}. \quad (2.14)$$

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Assume that $(f, \boldsymbol{\lambda}) \in \Lambda_{d,n}^+$ and that $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_n)$ is an n -updown $\boldsymbol{\lambda}$ -tableau. Set $\boldsymbol{\mu} = \mathcal{T}_{n-1}$ and $\mathcal{U} = (\mathcal{T}_1, \dots, \mathcal{T}_{n-1})$ as an updown $\boldsymbol{\mu}$ -tableau. Let $\boldsymbol{\theta}$ be the box that is addable to or removable from $\boldsymbol{\mu}$ to get $\boldsymbol{\lambda}$. For simplicity, we set $c_k := c(\mathcal{T}|k)$. By (2.13), we can rewrite $E_{\mathcal{T}}$ inductively as follows:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \frac{(X_n - b_1) \cdots (X_n - b_k)}{(c_n - b_1) \cdots (c_n - b_k)}, \quad (3.1)$$

where b_1, \dots, b_k are the contents of all boxes except $\boldsymbol{\theta}$, which can be addable to or removable from $\boldsymbol{\mu}$ to get a d -partition.

We denote by $\{\Lambda_1, \dots, \Lambda_h\}$ the set of all d -partitions obtained from $\boldsymbol{\mu}$ by adding a box or removing one. Set $\mathcal{T}_j := (\mathcal{T}_1, \dots, \mathcal{T}_{n-1}, \Lambda_j)$ for $1 \leq j \leq h$. Note that $\mathcal{T} \in \{\mathcal{T}_1, \dots, \mathcal{T}_h\}$. Since $\mathcal{B}_{d,n}$ is generic, hence it is semisimple. By [RuSi3, (4.16)] we have

$$E_{\mathcal{U}} = \sum_{j=1}^h E_{\mathcal{T}_j}. \quad (3.2)$$

The property (2.14) implies that the following rational function

$$E_{\mathcal{U}} \frac{u - c_n}{u - X_n} \quad (3.3)$$

is regular at $u = c_n$, and by (3.2), we get

$$E_{\mathcal{U}} \frac{u - c_n}{u - X_n} \Big|_{u=c_n} = E_{\mathcal{T}}. \quad (3.4)$$

For $1 \leq i \leq n-1$, we set

$$Q_i(u, v; c) := T_i + \frac{q - q^{-1}}{\rho^{-1}cuv - 1} + \frac{q - q^{-1}}{1 + qcuv} E_i. \quad (3.5)$$

Let $\phi_1(u) := \frac{cuX_1 - \rho}{u - X_1}$. For $k = 2, \dots, n$, we set

$$\begin{aligned} \phi_k(u_1, \dots, u_{k-1}, u) &:= Q_{k-1}(u_{k-1}, u; c) \phi_{k-1}(u_1, \dots, u_{k-2}, u) T_{k-1}(u_{k-1}, u) \\ &= Q_{k-1}(u_{k-1}, u; c) \cdots Q_1(u_1, u; c) \phi_1(u) T_1(u_1, u) \cdots T_{k-1}(u_{k-1}, u). \end{aligned} \quad (3.6)$$

From now on, we always set $c := -q^{-1}$. The following lemma is inspired by [IMOG2, Lemma 1] and can be proved similarly.

Lemma 3.1. *Assume that $n \geq 1$. We have*

$$E_{\mathcal{U}} \phi_n(c_1, \dots, c_{n-1}, u) \prod_{r=1}^{n-1} f(u, c_r)^{-1} = E_{\mathcal{U}} \frac{cuX_n - \rho}{u - X_n}. \quad (3.7)$$

Proof. We shall prove (3.7) by induction on n . For $n = 1$, the situation is trivial.

We set

$$\begin{aligned} \phi'_n(c_1, \dots, c_{n-1}, u) \\ = Q_{n-1}(c_{n-1}, u; c) \cdots Q_1(c_1, u; c) \phi_1(u) T_1(u, c_1)^{-1} \cdots T_{n-1}(u, c_{n-1})^{-1}. \end{aligned} \quad (3.8)$$

By (2.4) and (3.8), in order to show (3.7), it suffices to prove that

$$E_{\mathcal{U}} \phi'_n(c_1, \dots, c_{n-1}, u) = E_{\mathcal{U}} \frac{cuX_n - \rho}{u - X_n}. \quad (3.9)$$

By the induction hypothesis, it boils down to proving the following equality:

$$E_{\mathcal{U}} Q_{n-1}(c_{n-1}, u; c) \frac{cuX_{n-1} - \rho}{u - X_{n-1}} T_{n-1}(u, c_{n-1})^{-1} = E_{\mathcal{U}} \frac{cuX_n - \rho}{u - X_n}. \quad (3.10)$$

Since X_n commutes with $E_{\mathcal{U}}$, we can rewrite (3.10) as follows:

$$\begin{aligned} E_{\mathcal{U}}(u - X_n) Q_{n-1}(c_{n-1}, u; c) (cuX_{n-1} - \rho) \\ = E_{\mathcal{U}}(cuX_n - \rho) T_{n-1}(u, c_{n-1}) (u - X_{n-1}). \end{aligned} \quad (3.11)$$

By (2.3) and (3.5), the equality (3.11) becomes

$$\begin{aligned} E_{\mathcal{U}}(u - X_n) \left(T_{n-1} + \frac{q - q^{-1}}{\rho^{-1}cuc_{n-1} - 1} + \frac{q - q^{-1}}{1 + qcuc_{n-1}} E_{n-1} \right) (cuX_{n-1} - \rho) \\ = E_{\mathcal{U}}(cuX_n - \rho) \left(T_{n-1} + \frac{(q - q^{-1})u}{c_{n-1} - u} + \frac{(q - q^{-1})u}{u + \rho qc_{n-1}} E_{n-1} \right) (u - X_{n-1}). \end{aligned} \quad (3.12)$$

By definition, we have $T_{n-1}X_{n-1} = X_nT_{n-1} - (q - q^{-1})X_n + (q - q^{-1})X_nE_{n-1}$. Thus, we get that (3.12) is equivalent to

$$\begin{aligned} E_{\mathcal{U}}(u - X_n) & \left(cu(X_nT_{n-1} - (q - q^{-1})X_n + (q - q^{-1})X_nE_{n-1}) - \rho T_{n-1} \right. \\ & \quad \left. + (q - q^{-1})\rho + \frac{q - q^{-1}}{1 + qcuc_{n-1}} E_{n-1}(cuX_{n-1} - \rho) \right) \\ & = E_{\mathcal{U}}(cuX_n - \rho) \left(-X_nT_{n-1} + (q - q^{-1})X_n - (q - q^{-1})X_nE_{n-1} + uT_{n-1} \right. \\ & \quad \left. - (q - q^{-1})u + \frac{(q - q^{-1})u}{u + \rho qc_{n-1}} E_{n-1}(u - X_{n-1}) \right). \end{aligned} \quad (3.13)$$

Since we have

$$\begin{aligned} cu^2X_nT_{n-1} - cuX_n^2T_{n-1} - (q - q^{-1})cuX_n(u - X_n) - \rho(u - X_n)(T_{n-1} - (q - q^{-1})) \\ = -cuX_n^2T_{n-1} + \rho X_nT_{n-1} + (q - q^{-1})(cuX_n - \rho)X_n + u(cuX_n - \rho)(T_{n-1} - (q - q^{-1})), \end{aligned}$$

it is easy to see that the equality (3.13) comes down to the following equality:

$$\begin{aligned} E_{\mathcal{U}}(u - X_n) & \left(cuX_nE_{n-1} + \frac{1}{1 + qcuc_{n-1}} E_{n-1}(cuX_{n-1} - \rho) \right) \\ & = E_{\mathcal{U}}(cuX_n - \rho) \left(-X_nE_{n-1} + \frac{u}{u + \rho qc_{n-1}} E_{n-1}(u - X_{n-1}) \right). \end{aligned} \quad (3.14)$$

By definition, we have $E_{\mathcal{U}}X_{n-1} = c_{n-1}E_{\mathcal{U}}$. Hence, we get $E_{\mathcal{U}}X_nE_{n-1} = \frac{1}{c_{n-1}}E_{\mathcal{U}}E_{n-1}$ by (2.2). According to this, by comparing the coefficients of the terms involving $E_{\mathcal{U}}E_{n-1}X_{n-1}$, we see that it suffices to show that

$$\frac{1}{1 + qcuc_{n-1}} \cdot \frac{cu^2c_{n-1} - cu}{c_{n-1}} = \frac{u}{u + \rho qc_{n-1}} \cdot \frac{\rho c_{n-1} - cu}{c_{n-1}}. \quad (3.15)$$

By comparing the coefficients of the terms involving $E_{\mathcal{U}}E_{n-1}$, it suffices to show that

$$\frac{cu^2 - \rho}{c_{n-1}} + \frac{1}{1 + qcuc_{n-1}} \cdot \frac{\rho - \rho uc_{n-1}}{c_{n-1}} = \frac{u}{u + \rho qc_{n-1}} \cdot \frac{cu^2 - \rho uc_{n-1}}{c_{n-1}}. \quad (3.16)$$

Noting that $c = -q^{-1}$, it is easy to verify that (3.15) and (3.16) are true. Thus, (3.14) holds. The lemma is proved. \square

Let $\bar{\phi}_1(u) := (u - v_1) \cdots (u - v_d) \frac{cuX_1 - \rho}{u - X_1}$. For $k = 2, \dots, n$, we set

$$\begin{aligned} \bar{\phi}_k(u_1, \dots, u_{k-1}, u) & := Q_{k-1}(u_{k-1}, u; c) \bar{\phi}_{k-1}(u_1, \dots, u_{k-2}, u) T_{k-1}(u_{k-1}, u) \\ & = Q_{k-1}(u_{k-1}, u; c) \cdots Q_1(u_1, u; c) \bar{\phi}_1(u) T_1(u_1, u) \cdots T_{k-1}(u_{k-1}, u). \end{aligned} \quad (3.17)$$

We also define the following rational function:

$$\Phi(u_1, \dots, u_n) := \bar{\phi}_1(u_1) \cdots \bar{\phi}_{n-1}(u_1, \dots, u_{n-1}) \bar{\phi}_n(u_1, \dots, u_n). \quad (3.18)$$

Recall that the integers p_1, \dots, p_n associated to \mathcal{T} have been defined as in (2.10) or (2.11).

Now we can state the main result of this paper.

Theorem 3.2. *The idempotent $E_{\mathcal{T}}$ of $\mathcal{B}_{d,n}$ corresponding to an n -updown λ -tableau \mathcal{T} can be derived by the following consecutive evaluations:*

$$E_{\mathcal{T}} = \frac{1}{f(\mathcal{T})} \left(\prod_{k=1}^n \frac{(u_k - c_k)^{p_k}}{cu_k c_k - \rho} \right) \Phi(u_1, \dots, u_n) \Big|_{u_1=c_1} \cdots \Big|_{u_n=c_n}. \quad (3.19)$$

Proof. We shall prove the theorem by induction on n . For $n = 1$, we have $p_1 = 0$ by Proposition 2.2. Thus, we get that the right-hand side of (3.19) is equal to

$$\begin{aligned} & \frac{1}{f(\mathcal{T})} \frac{(u_1 - v_1) \cdots (u_1 - v_d)}{cu_1 c_1 - \rho} \frac{cu_1 X_1 - \rho}{u_1 - X_1} \Big|_{u_1=c_1} \\ &= \frac{1}{f(\mathcal{T})} \frac{(u_1 - v_1) \cdots (u_1 - v_d)}{u_1 - c_1} \frac{u_1 - c_1}{cu_1 c_1 - \rho} \frac{cu_1 X_1 - \rho}{u_1 - X_1} \Big|_{u_1=c_1}. \end{aligned} \quad (3.20)$$

Moreover, by (2.12), we have

$$f(\mathcal{T}) = \prod_{1 \leq k \leq d; v_k \neq c_1} (c_1 - v_k).$$

Therefore, it is easy to see that (3.20) is equal to $E_{\mathcal{T}}$ by (2.14) and (3.4).

For $n \geq 2$, by the induction hypothesis we can write the right-hand side of (3.19) as follows:

$$\frac{f(\mathcal{U})}{f(\mathcal{T})} \frac{(u_n - c_n)^{p_n}}{cu_n c_n - \rho} E_{\mathcal{U}} \bar{\phi}_n(c_1, \dots, c_{n-1}, u_n) \Big|_{u_n=c_n}. \quad (3.21)$$

Note that $\bar{\phi}_n(c_1, \dots, c_{n-1}, u_n) = (u_n - v_1) \cdots (u_n - v_d) \phi_n(c_1, \dots, c_{n-1}, u_n)$. By (3.7), we can rewrite the expression (3.21) as

$$\frac{f(\mathcal{U})}{f(\mathcal{T})} \frac{(u_n - c_n)^{p_n}}{cu_n c_n - \rho} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} f(u_n, c_r) E_{\mathcal{U}} \frac{cu_n X_n - \rho}{u_n - X_n} \Big|_{u_n=c_n}. \quad (3.22)$$

By (2.12), we see that

$$\begin{aligned} & \frac{f(\mathcal{U})}{f(\mathcal{T})} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} f(u_n, c_r) (u_n - c_n)^{p_n-1} \\ &= \frac{f(\mathcal{U})}{f(\mathcal{T})} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} \frac{(u_n - q^2 c_r)(u_n - q^{-2} c_r)}{(u_n - c_r)^2} (u_n - c_n)^{p_n-1} \end{aligned}$$

is regular at $u_n = c_n$ and is equal to 1. Thus, the expression (3.22) equals

$$E_{\mathcal{U}} \frac{u_n - c_n}{u_n - X_n} \frac{cu_n X_n - \rho}{cu_n c_n - \rho} \Big|_{u_n=c_n}. \quad (3.23)$$

By (3.4), we see that (3.23) is equal to

$$E_{\mathcal{T}} \frac{cu_n X_n - \rho}{cu_n c_n - \rho} \Big|_{u_n=c_n}. \quad (3.24)$$

By (2.14), we have $E_{\mathcal{T}} X_n = c_n E_{\mathcal{T}}$. Thus, we get that the expression (3.24), that is, the right-hand side of (3.19) equals $E_{\mathcal{T}}$. \square

Remark 3.3. Let $\mathcal{H}_{d,n}$ be the cyclotomic Hecke algebra defined in [AK]. It has been proved in [RuXu, Proposition 4.1] that $\mathcal{H}_{d,n}$ is isomorphic to the quotient of $\mathcal{B}_{d,n}$ by the two-sided ideal generated by all E_i . In the process of taking quotient, the parameter ρ disappears; however, the parameter c is reserved and can be arbitrary. If we replace the $T_i(u, v)$, $Q_i(u, v; c)$, $\phi_1(u)$ in (3.6) with

$$\overline{T}_i(u, v) = T_i + \frac{(q - q^{-1})u}{v - u}, \quad \overline{Q}_i(u, v; c) := T_i + \frac{q - q^{-1}}{cuv - 1}, \quad \psi_1(u) := \frac{cuX_1 - 1}{u - X_1},$$

it is easy to see that the analogue of Lemma 3.1 holds.

Let $\overline{\psi}_1(u) := (u - v_1) \cdots (u - v_d) \frac{cuX_1 - 1}{u - X_1}$, and for $k = 2, \dots, n$, set

$$\overline{\psi}_k(u_1, \dots, u_{k-1}, u) := \overline{Q}_{k-1}(u_{k-1}, u; c) \overline{\psi}_{k-1}(u_1, \dots, u_{k-2}, u) \overline{T}_{k-1}(u_{k-1}, u).$$

We also define a rational function by

$$\Upsilon(u_1, \dots, u_n) := \overline{\psi}_1(u_1) \cdots \overline{\psi}_{n-1}(u_1, \dots, u_{n-1}) \overline{\psi}_n(u_1, \dots, u_n).$$

Then it is easy to see that the analogue of Theorem 3.2 is true. Thus, we get a one-parameter family of the fusion procedures for cyclotomic Hecke algebras, generalizing the results obtained in [OgPA2].

4. APPENDIX. FUSION PROCEDURE FOR CYCLOTOMIC NAZAROV-WENZL ALGEBRAS

When studying the representations of Brauer algebras, Nazarov [Na1] introduced a class of infinite dimensional algebras under the name affine Wenzl algebras. In order to study finite dimensional irreducible representations of affine Wenzl algebras, Ariki, Mathas and Rui [AMR] defined the finite dimensional quotients of them, known as the cyclotomic Nazarov-Wenzl algebras. Cyclotomic Nazarov-Wenzl algebras are related to degenerate cyclotomic Hecke algebras just in the same way that cyclotomic BMW algebras are connected with cyclotomic Hecke algebras. Cyclotomic Nazarov-Wenzl algebras have been studied by many authors; see [Go3-4, RuSi1-2, Xu] and so on.

4.1. Cyclotomic Nazarov-Wenzl algebras.

Definition 4.1. Suppose that \mathbb{K} is an algebraically closed field containing ω_j ($0 \leq j \leq d - 1$), v_i ($1 \leq i \leq d$), and the invertible element 2.

Fix $n \geq 1$. The cyclotomic Nazarov-Wenzl algebra $\mathcal{W}_{d,n}$ is the \mathbb{K} -algebra generated by the elements S_i, E_i ($1 \leq i \leq n - 1$) and X_j ($1 \leq j \leq n$) satisfying the following relations:

- (1) (Involutions) $S_i^2 = 1$ for $1 \leq i \leq n - 1$.
- (2) (Idempotent relations) $E_i^2 = \omega_0 E_i$ for $1 \leq i \leq n - 1$.
- (3) (Affine braid relations)
 - (a) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$ and $S_i S_j = S_j S_i$ if $|i - j| \geq 2$.
 - (b) $S_i X_j = X_j S_i$ if $j \neq i, i + 1$.
- (4) (Tangle relations)
 - (a) $E_i E_{i \pm 1} E_i = E_i$.
 - (b) $S_i S_{i \pm 1} E_i = E_{i \pm 1} E_i$ and $E_i S_{i \pm 1} S_i = E_i E_{i \pm 1}$.
 - (c) For $1 \leq k \leq d - 1$, $E_1 X_1^k E_1 = \omega_k E_1$.
- (5) (Untwisting relations) $S_i E_i = E_i S_i = E_i$ for $1 \leq i \leq n - 1$.

- (6) (Skein relations) $S_i X_i - X_{i+1} S_i = E_i - 1$ for $1 \leq i \leq n-1$.
- (7) (Anti-symmetry relations) $E_i(X_i + X_{i+1}) = (X_i + X_{i+1})E_i = 0$ for $1 \leq i \leq n-1$.
- (8) (Commutative relations)
 - (a) $S_i E_j = E_j S_i$ and $E_i E_j = E_j E_i$ if $|i - j| \geq 2$.
 - (b) $E_i X_j = X_j E_i$ if $j \neq i, i+1$.
 - (c) $X_i X_j = X_j X_i$ for $1 \leq i, j \leq n$.
- (9) (Cyclotomic relation) $(X_1 - v_1)(X_1 - v_2) \cdots (X_1 - v_d) = 0$.

We define the following elements:

$$S_i(u, v) = S_i + \frac{1}{v - u} - \frac{1}{v - u + \frac{\omega_0}{2} - 1} E_i \quad \text{for } 1 \leq i \leq n-1. \quad (4.1)$$

By using the fact that $E_i^2 = \omega_0 E_i$, we can easily get

$$S_i(u, v) S_i(v, u) = g(u, v) \quad \text{for } 1 \leq i \leq n-1, \quad (4.2)$$

where

$$g(u, v) = g(v, u) = \frac{(u - v + 1)(u - v - 1)}{(u - v)^2}. \quad (4.3)$$

4.2. Combinatorics. Suppose that $(f, \lambda) \in \Lambda_{d,n}^+$ and $\mathfrak{s} = (\mathfrak{s}_1, \dots, \mathfrak{s}_n) \in \mathcal{T}_n^{ud}(\lambda)$. We can define the integers $d_k^s, \bar{d}_k^s, g_k^s, \bar{g}_k^s$ and some integers p_1, \dots, p_n associated to \mathfrak{s} in exactly the same way as those related to some \mathcal{T} defined in Subsection 2.2. We shall follow the notations and only emphasize the differences.

Set

$$c(\mathfrak{s}|k) = \begin{cases} v_s + j - i & \text{if } \mathfrak{s}_k = \mathfrak{s}_{k-1} \cup ((i, j), s), \\ -v_s + i - j & \text{if } \mathfrak{s}_{k-1} = \mathfrak{s}_k \cup ((i, j), s). \end{cases} \quad (4.4)$$

Given a box $\beta = ((i, j), s)$, we define the content of it by

$$c(\mathcal{U}|\beta) = \begin{cases} v_s + j - i & \text{if } \beta \text{ is an addable box of } \mathfrak{s}, \\ -v_s + i - j & \text{if } \beta \text{ is a removable box of } \mathfrak{s}. \end{cases} \quad (4.5)$$

Assume that $(f, \lambda) \in \Lambda_{d,n}^+$, $\mathfrak{t} = (\mathfrak{t}_1, \dots, \mathfrak{t}_n)$ is an n -updown λ -tableau and that $\mathfrak{u} = (\mathfrak{t}_1, \dots, \mathfrak{t}_{n-1})$. We then define the element $g(\mathfrak{t})$ inductively by

$$g(\mathfrak{t}) = g(\mathfrak{u})\psi(\mathfrak{u}, \mathfrak{t}), \quad (4.6)$$

where

$$\psi(\mathfrak{u}, \mathfrak{t}) = \prod_{\substack{k \neq k_n \\ k \in \mathbb{Z}}} (k_n - k)^{g_k^{s_n}} \prod_{\substack{1 \leq t \leq d; t \neq s_n \\ k \in \mathbb{Z}}} (v_{s_n} - v_t + k_n - k)^{g_k^t} \prod_{\substack{1 \leq r \leq d \\ k \in \mathbb{Z}}} (v_{s_n} + v_r + k_n + k)^{\bar{g}_k^r}$$

if \mathfrak{t}_n is obtained from \mathfrak{t}_{n-1} by adding a box $((i_n, j_n), s_n)$, where $k_n = j_n - i_n$;

$$\psi(\mathfrak{u}, \mathfrak{t}) = \prod_{\substack{k \neq k'_n \\ k \in \mathbb{Z}}} (-k'_n + k)^{\bar{g}_k^{s'_n}} \prod_{\substack{1 \leq t \leq d; t \neq s'_n \\ k \in \mathbb{Z}}} (-v_{s'_n} + v_t - k'_n + k)^{\bar{g}_k^t} \prod_{\substack{1 \leq r \leq d \\ k \in \mathbb{Z}}} (-v_{s'_n} - v_r - k'_n - k)^{g_k^r}$$

if \mathfrak{t}_n is obtained from \mathfrak{t}_{n-1} by removing a box $((i'_n, j'_n), s'_n)$, where $k'_n = j'_n - i'_n$.

The following proposition is inspired by [IM, Proposition 3.3] and can be proved similarly.

Proposition 4.2. *If λ is a d -partition of n and $\mathbf{t} = (t_1, \dots, t_n)$ is an n -updown λ -tableau, then p_1, \dots, p_n are all equal to zero, and $g(\mathbf{t})$ is exactly equal to $\Theta_\lambda(Q)^{-1}$ defined in [ZL, (3.2)] when $d = m$ and $v_s = q_s$ for $1 \leq s \leq m$.*

4.3. Idempotents of $\mathscr{W}_{d,n}$. Following [AMR, Definition 4.3], we say that $\mathscr{W}_{d,n}$ is generic if the parameters v_i , $1 \leq i \leq d$, satisfy the conditions (1) the characteristic p of \mathbb{K} satisfies $p = 0$ or $p > 2n$; (2) $|r| \geq 2n$ whenever there exists $r \in \mathbb{Z}$ such that either $v_i \pm v_j = r$ and $i \neq j$, or $2v_i = r$. Following [Go3, Definition 4.2], we say that $\mathscr{W}_{d,n}$ is admissible if the set $\{E_1, E_1 X_1, \dots, E_1 X_1^{d-1}\}$ is linearly independent in $\mathscr{B}_{d,2}$. It has been proved by Goodman [Go3, Theorem 5.2] that this admissible condition coincides with the **u**-admissible condition defined in [AMR, Definition 3.6].

From now on, we always assume that $\mathscr{W}_{d,n}$ is generic and admissible. Thus, by [AMR, Lemma 4.4], we have $\mathfrak{s} = \mathfrak{t}$ if and only if $c(\mathfrak{s}|k) = c(\mathfrak{t}|k)$ for all $1 \leq k \leq n$. Therefore, the set $\{X_1, \dots, X_n\}$, as a family of JM-elements for $\mathscr{W}_{d,n}$ in the abstract sense defined in [Ma, Definition 2.4], satisfies the separation condition associated to the cellular basis of $\mathscr{W}_{d,n}$ constructed in [AMR, Theorem 7.17]. In particular, we can construct the primitive idempotents of $\mathscr{W}_{d,n}$ following the arguments in [Ma, Section 3].

For each $1 \leq k \leq n$, we define the following set:

$$\mathscr{R}(k) := \{c(\mathfrak{s}|k) \mid \mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda) \text{ for some } (f, \lambda) \in \Lambda_{d,n}^+\}.$$

Suppose that $(f, \lambda) \in \Lambda_{d,n}^+$ and $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. We set

$$E_{\mathfrak{t}} = \prod_{k=1}^n \left(\prod_{\substack{a \in \mathscr{R}(k) \\ a \neq c(\mathfrak{t}|k)}} \frac{X_k - a}{c(\mathfrak{t}|k) - a} \right). \quad (4.7)$$

By standard arguments in [Ma, Section 3], the elements $\{E_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda) \text{ for some } (f, \lambda) \in \Lambda_{d,n}^+\}$ form a complete set of pairwise orthogonal primitive idempotents of $\mathscr{W}_{d,n}$. Moreover, the elements X_1, \dots, X_n generate a maximal commutative subalgebra of $\mathscr{W}_{d,n}$. We also have

$$X_k E_{\mathfrak{t}} = E_{\mathfrak{t}} X_k = c(\mathfrak{t}|k) E_{\mathfrak{t}}. \quad (4.8)$$

4.4. Fusion procedure for cyclotomic Nazarov-Wenzl algebras. Assume that $(f, \lambda) \in \Lambda_{d,n}^+$ and that $\mathfrak{t} = (t_1, \dots, t_n)$ is an n -updown λ -tableau. Set $\mu = \mathfrak{t}_{n-1}$ and $\mathfrak{u} = (t_1, \dots, t_{n-1})$ as an updown μ -tableau. Let θ be the box that is addable to or removable from μ to get λ . For simplicity, we set $c_k := c(\mathfrak{t}|k)$. By (4.7), we can rewrite $E_{\mathfrak{t}}$ inductively as follows:

$$E_{\mathfrak{t}} = E_{\mathfrak{t}} \frac{(X_n - a_1) \cdots (X_n - a_k)}{(c_n - a_1) \cdots (c_n - a_k)}, \quad (4.9)$$

where a_1, \dots, a_k are the contents of all boxes except θ , which can be addable to or removable from μ to get a d -partition.

We denote by $\{\Delta_1, \dots, \Delta_e\}$ the set of all d -partitions obtained from μ by adding a box or removing one. Set $\mathscr{S}_j := (t_1, \dots, t_{n-1}, \Delta_j)$ for $1 \leq j \leq e$. Note that $\mathfrak{t} \in \{\mathscr{S}_1, \dots, \mathscr{S}_e\}$.

Since $\mathcal{W}_{d,n}$ is generic, hence it is semisimple. By [AMR, Theorem 5.3 a)] we have

$$E_u = \sum_{j=1}^e E_{\mathcal{J}_j}. \quad (4.10)$$

The equality (4.8) implies that the following rational function

$$E_u \frac{u - c_n}{u - X_n} \quad (4.11)$$

is regular at $u = c_n$, and by (4.10), we get

$$E_u \frac{u - c_n}{u - X_n} \Big|_{u=c_n} = E_t. \quad (4.12)$$

For $1 \leq i \leq n-1$, we set

$$R_i(u, v; c) := S_i + \frac{1}{u + v + c} - \frac{1}{u + v} E_i. \quad (4.13)$$

Let $\varphi_1(u) := \frac{u + X_1 + c}{u - X_1}$. For $k = 2, \dots, n$, we set

$$\begin{aligned} \varphi_k(u_1, \dots, u_{k-1}, u) &:= R_{k-1}(u_{k-1}, u; c) \varphi_{k-1}(u_1, \dots, u_{k-2}, u) S_{k-1}(u_{k-1}, u) \\ &= R_{k-1}(u_{k-1}, u; c) \cdots R_1(u_1, u; c) \varphi_1(u) S_1(u_1, u) \cdots S_{k-1}(u_{k-1}, u). \end{aligned} \quad (4.14)$$

From now on, we always set $c := 1 - \frac{\omega_0}{2}$. The following lemma is inspired by [IMOG2, Lemma 1] and can be proved similarly.

Lemma 4.3. *Assume that $n \geq 1$. We have*

$$E_u \varphi_n(c_1, \dots, c_{n-1}, u) \prod_{r=1}^{n-1} g(u, c_r)^{-1} = E_u \frac{u + X_n + c}{u - X_n}. \quad (4.15)$$

Proof. We shall prove (4.15) by induction on n . For $n = 1$, the situation is trivial.

We set

$$\begin{aligned} \varphi'_n(c_1, \dots, c_{n-1}, u) \\ = R_{n-1}(c_{n-1}, u; c) \cdots R_1(c_1, u; c) \varphi_1(u) S_1(u, c_1)^{-1} \cdots S_{n-1}(u, c_{n-1})^{-1}. \end{aligned} \quad (4.16)$$

By (4.2) and (4.16), in order to show (4.15), it suffices to prove that

$$E_u \varphi'_n(c_1, \dots, c_{n-1}, u) = E_u \frac{u + X_n + c}{u - X_n}. \quad (4.17)$$

By the induction hypothesis, it boils down to proving the following equality:

$$E_u R_{n-1}(c_{n-1}, u; c) \frac{u + X_{n-1} + c}{u - X_{n-1}} S_{n-1}(u, c_{n-1})^{-1} = E_u \frac{u + X_n + c}{u - X_n}. \quad (4.18)$$

Since X_n commutes with E_u , we can rewrite (4.18) as follows:

$$\begin{aligned} E_u (u - X_n) R_{n-1}(c_{n-1}, u; c) (u + X_{n-1} + c) \\ = E_u (u + X_n + c) S_{n-1}(u, c_{n-1}) (u - X_{n-1}). \end{aligned} \quad (4.19)$$

By (4.1) and (4.13), the equality (4.19) becomes

$$\begin{aligned} E_u(u - X_n) \left(S_{n-1} + \frac{1}{c_{n-1} + u + c} - \frac{1}{c_{n-1} + u} E_{n-1} \right) (u + X_{n-1} + c) \\ = E_u(u + X_n + c) \left(S_{n-1} + \frac{1}{c_{n-1} - u} - \frac{1}{c_{n-1} - u + \frac{\omega_0}{2} - 1} E_{n-1} \right) (u - X_{n-1}). \end{aligned} \quad (4.20)$$

By definition, we have $S_{n-1}X_{n-1} = X_nS_{n-1} + E_{n-1} - 1$. Thus, we get that (4.20) is equivalent to

$$\begin{aligned} E_u(u - X_n) \left(uS_{n-1} + (X_nS_{n-1} + E_{n-1} - 1) + cS_{n-1} + 1 \right. \\ \left. - \frac{1}{c_{n-1} + u} E_{n-1}(u + X_{n-1} + c) \right) \\ = E_u(u + X_n + c) \left(uS_{n-1} - (X_nS_{n-1} + E_{n-1} - 1) - 1 \right. \\ \left. - \frac{1}{c_{n-1} - u + \frac{\omega_0}{2} - 1} E_{n-1}(u - X_{n-1}) \right). \end{aligned} \quad (4.21)$$

It is easy to see that the equality (4.21) comes down to the following equality:

$$\begin{aligned} (c + 2u)E_u - E_u(u - X_n) \frac{1}{c_{n-1} + u} E_{n-1}(u + X_{n-1} + c) \\ = -E_u(u + X_n + c) \frac{1}{c_{n-1} - u + \frac{\omega_0}{2} - 1} E_{n-1}(u - X_{n-1}). \end{aligned} \quad (4.22)$$

By definition, we have $E_uX_{n-1} = c_{n-1}E_u$. Hence, we get $E_uX_nE_{n-1} = -c_{n-1}E_uE_{n-1}$ by definition. According to this, by comparing the coefficients of the terms involving $E_uE_{n-1}X_{n-1}$, we see that it suffices to show that

$$\frac{-u - c_{n-1}}{c_{n-1} + u} = \frac{u - c_{n-1} + c}{c_{n-1} - u + \frac{\omega_0}{2} - 1}. \quad (4.23)$$

By comparing the coefficients of the terms involving E_uE_{n-1} , it suffices to show that

$$(c + 2u) + \frac{-(c + u)(c_{n-1} + u)}{c_{n-1} + u} = \frac{u(-u + c_{n-1} - c)}{c_{n-1} - u + \frac{\omega_0}{2} - 1}. \quad (4.24)$$

Noting that $c = 1 - \frac{\omega_0}{2}$, it is easy to verify that (4.23) and (4.24) are true. Thus, (4.22) holds. The lemma is proved. \square

Let $\overline{\varphi}_1(u) := (u - v_1) \cdots (u - v_d) \frac{u + X_1 + c}{u - X_1}$. For $k = 2, \dots, n$, we set

$$\begin{aligned} \overline{\varphi}_k(u_1, \dots, u_{k-1}, u) &:= R_{k-1}(u_{k-1}, u; c) \overline{\varphi}_{k-1}(u_1, \dots, u_{k-2}, u) S_{k-1}(u_{k-1}, u) \\ &= R_{k-1}(u_{k-1}, u; c) \cdots R_1(u_1, u; c) \overline{\varphi}_1(u) S_1(u_1, u) \cdots S_{k-1}(u_{k-1}, u). \end{aligned} \quad (4.25)$$

We also define the following rational function:

$$\Psi(u_1, \dots, u_n) := \overline{\varphi}_1(u_1) \cdots \overline{\varphi}_{n-1}(u_1, \dots, u_{n-1}) \overline{\varphi}_n(u_1, \dots, u_n). \quad (4.26)$$

Recall that the integers p_1, \dots, p_n associated to \mathfrak{t} have been defined as in (2.10) or (2.11).

Now we can state the main result of this paper.

Theorem 4.4. *The idempotent E_t of $\mathcal{W}_{d,n}$ corresponding to an n -updown λ -tableau t can be derived by the following consecutive evaluations:*

$$E_t = \frac{1}{g(t)} \left(\prod_{k=1}^n \frac{(u_k - c_k)^{p_k}}{u_k + c_k + c} \right) \Psi(u_1, \dots, u_n) \Big|_{u_1=c_1} \cdots \Big|_{u_n=c_n}. \quad (4.27)$$

Proof. We shall prove the theorem by induction on n . For $n = 1$, we have $p_1 = 0$ by Proposition 4.2. Thus, we get that the right-hand side of (4.27) is equal to

$$\begin{aligned} & \frac{1}{g(t)} \frac{(u_1 - v_1) \cdots (u_1 - v_d)}{u_1 + c_1 + c} \frac{u_1 + X_1 + c}{u_1 - X_1} \Big|_{u_1=c_1} \\ &= \frac{1}{g(t)} \frac{(u_1 - v_1) \cdots (u_1 - v_d)}{u_1 - c_1} \frac{u_1 - c_1}{u_1 + c_1 + c} \frac{u_1 + X_1 + c}{u_1 - X_1} \Big|_{u_1=c_1}. \end{aligned} \quad (4.28)$$

Moreover, by (4.6), we have

$$g(t) = \prod_{1 \leq k \leq d; v_k \neq c_1} (c_1 - v_k).$$

Therefore, it is easy to see that (4.28) is equal to E_t by (4.8) and (4.12).

For $n \geq 2$, by the induction hypothesis we can write the right-hand side of (4.27) as follows:

$$\frac{g(u)}{g(t)} \frac{(u_n - c_n)^{p_n}}{u_n + c_n + c} E_u \bar{\varphi}_n(c_1, \dots, c_{n-1}, u_n) \Big|_{u_n=c_n}. \quad (4.29)$$

Note that $\bar{\varphi}_n(c_1, \dots, c_{n-1}, u_n) = (u_n - v_1) \cdots (u_n - v_d) \varphi_n(c_1, \dots, c_{n-1}, u_n)$. By (4.15), we can rewrite the expression (4.29) as

$$\frac{g(u)}{g(t)} \frac{(u_n - c_n)^{p_n}}{u_n + c_n + c} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} g(u_n, c_r) E_u \frac{u_n + X_n + c}{u_n - X_n} \Big|_{u_n=c_n}. \quad (4.30)$$

By (4.6), we see that

$$\begin{aligned} & \frac{g(u)}{g(t)} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} g(u_n, c_r) (u_n - c_n)^{p_n-1} \\ &= \frac{g(u)}{g(t)} (u_n - v_1) \cdots (u_n - v_d) \prod_{r=1}^{n-1} \frac{(u_n - c_r + 1)(u_n - c_r - 1)}{(u_n - c_r)^2} (u_n - c_n)^{p_n-1} \end{aligned}$$

is regular at $u_n = c_n$ and is equal to 1. Thus, the expression (4.30) equals

$$E_u \frac{u_n - c_n}{u_n - X_n} \frac{u_n + X_n + c}{u_n + c_n + c} \Big|_{u_n=c_n}. \quad (4.31)$$

By (4.12), we see that (4.31) is equal to

$$E_t \frac{u_n + X_n + c}{u_n + c_n + c} \Big|_{u_n=c_n}. \quad (4.32)$$

By (4.8), we have $E_t X_n = c_n E_t$. Thus, we get that the expression (4.32), that is, the right-hand side of (4.27) equals E_t . \square

Remark 4.5. Let $\mathcal{D}_{d,n}$ be the degenerate cyclotomic Hecke algebra. It has been proved in [AMR, Proposition 7.2] that $\mathcal{D}_{d,n}$ is isomorphic to the quotient of $\mathcal{W}_{d,n}$ by the two-sided ideal generated by all E_i . In the process of taking quotient, the parameter ω_0 disappears; however, the parameter c is reserved and can be arbitrary. If we replace the $S_i(u, v)$, $R_i(u, v; c)$, $\varphi_1(u)$ in (4.14) with

$$\bar{S}_i(u, v) = S_i + \frac{1}{v - u}, \quad \bar{R}_i(u, v; c) := S_i + \frac{1}{u + v + c}, \quad \chi_1(u) := \frac{u + X_1 + c}{u - X_1},$$

it is easy to see that the analogue of Lemma 4.3 holds.

Let $\bar{\chi}_1(u) := (u - v_1) \cdots (u - v_d) \frac{u + X_1 + c}{u - X_1}$, and for $k = 2, \dots, n$, set

$$\bar{\chi}_k(u_1, \dots, u_{k-1}, u) := \bar{R}_{k-1}(u_{k-1}, u; c) \bar{\chi}_{k-1}(u_1, \dots, u_{k-2}, u) \bar{S}_{k-1}(u_{k-1}, u).$$

We also define a rational function by

$$\Omega(u_1, \dots, u_n) := \bar{\chi}_1(u_1) \cdots \bar{\chi}_{n-1}(u_1, \dots, u_{n-1}) \bar{\chi}_n(u_1, \dots, u_n).$$

Then it is easy to see that the analogue of Theorem 4.4 is true. Thus, we get a one-parameter family of the fusion procedures for degenerate cyclotomic Hecke algebras, generalizing the results obtained in [ZL].

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